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SG PSEUDO-DIFFERENTIAL OPERATORS AND WEAK HYPERBOLICITY

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Abstract. We consider a class of pseudo-differential operators globally defined in $\mathbb{R}^n$. For them we discuss trace functionals, distribution of eigenvalues, essential spectrum and weak hyperbolicity.

1. Introduction

Pseudo-differential operators of SG type (Symbols of Global type) represent important examples of operators on non-compact manifolds. In fact, starting from the class $S^{\mu,\rho}$ of the symbols $a(x,\xi)$ satisfying in $\mathbb{R}^n \times \mathbb{R}^n$ estimates of the form

$$|D_x^\alpha D_\xi^\beta a(x,\xi)| \le C_{\alpha\beta}\langle x\rangle^{\rho-|\alpha|}\langle \xi\rangle^{\mu-|\beta|}, \tag{1}$$

where $\langle x\rangle = (1+|x|^2)^{1/2}$, $\langle \xi\rangle = (1+|\xi|^2)^{1/2}$, $\rho \in \mathbb{R}$, $\mu \in \mathbb{R}$, with $C_{\alpha\beta}$ independent of $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, one defines pseudo-differential operators having a suitable symbolic calculus in $\mathcal{S}(\mathbb{R}^n)$, $\mathcal{S}'(\mathbb{R}^n)$, see Parenti [25], Shubin [29], Cordes [5]. This class of operators can be then transferred in a natural way to non-compact C^∞-manifolds with exits to infinity, see Schrohe [26].

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A relevant subclass of $S^{\mu,\rho}$ is given by $S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$, the set of the classical symbols, admitting asymptotic expansions in x and ξ-homogeneous functions, see Schulze [27], Melrose [20],[21].

In the present paper, limiting attention to the $\mathbb{R}^n$ framework, we shall survey some recent contributions on SG operators and present also new results.

Namely, in Section 2 we report on the results of Nicola [23] concerning trace functionals, cf. Wodzicki [30]. Moreover we recall the related theorem of Maniccia [18], Maniccia and Panarese [19] about Weyl formula for the asymptotic distribution of the eigenvalues of self-adjoint SG operators. We address to Nicola [24] for the computation of the K-theory groups for operators with symbol in $S^{0,0}_{\mathrm{cl}(\xi,x)}$.

A new result is presented in Section 3, concerning location of the essential spectrum of SG operators.

Section 4 is devoted to SG hyperbolic equations. The basic definitions and the study of the strictly hyperbolic case were already in Cordes [5], treating also some weakly hyperbolic systems. A general approach to SG hyperbolic operators with constant multiplicity has been recently drawn by Coriasco, using SG Fourier integral operators, see Coriasco [6],[7], Coriasco-Rodino [9], Coriasco-Panarese [8], Coriasco-Maniccia [10]. Namely, we shall recall the result of Coriasco [7] about constant multiplicity under Levi condition, novelty in the present paper being a new definition of hyperbolicity for classical SG operators. We report also on the contribution of Cappiello [1],[2] concerning well-posedness of the Cauchy problem in the spaces of Gelfand-Shilov [13], under weakened Levi conditions.

2. Trace functionals

We begin by recalling some definitions and properties of the algebras of pseudo-differential operators we shall study, for details we refer to Schulze [27], Melrose [20],[21]; for the proofs of the theorems of this section see [23].

Let $\mathbb{S}^n_+ = \mathbb{S}^n \cap \{x = (x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} \geq 0\}$ and consider the so-called *radial compactification map*

$$\mathbb{R}^n \ni z \xrightarrow{\mathrm{RC}} (z/\langle z\rangle, 1/\langle z\rangle) \in \mathbb{S}^n_+ \subset \mathbb{R}^{n+1},$$

and $\mathrm{RC}^2 := \mathrm{RC} \times \mathrm{RC} : \mathbb{R}^{2n} \to \mathbb{S}^n_+ \times \mathbb{S}^n_+$.

Fix a real function $0 \leq r \in C^\infty(\mathbb{S}^n_+)$ such that

$$\mathbb{S}^{n-1} \equiv \partial\mathbb{S}^n_+ = \{r = 0\}$$

and $|z| = \mathrm{RC}^*(r)$ for large $|z|$.

We set $r_\psi := 1 \otimes r$ and $r_e := r \otimes 1$ on $\mathbb{S}^n_+ \times \mathbb{S}^n_+$.

Definition 1. *Let $\mu \in \mathbb{R}, \rho \in \mathbb{R}$. We define the symbol classes*

$$S^{\mu,\rho}_{\mathrm{cl}(\xi,x)} := (\mathrm{RC}^2)^* \left(r_\psi^{-\mu} r_e^{-\rho} C^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+) \right)$$

and the spaces $L^{\mu,\rho}_{\mathrm{cl}} := \mathrm{Op}^{\mathrm{w}} \left(S^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \right)$ of the corresponding pseudo-differential operators: if $a(x,\xi) \in S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ one sets as standard in the Weyl sense

$$Au(x) = a^{\mathrm{w}} u(x) = (2\pi)^{-n} \int e^{i(x-y)\xi} a\left(\frac{x+y}{2}, \xi \right) u(y)\, dy\, d\xi.$$

Later in Section 4, we shall also refer to the left-quantization:

$$a(x,D)u(x) = (2\pi)^{-n} \int e^{ix\xi} a(x,\xi) \hat{u}(\xi)\, d\xi.$$

Remark 2. Symbols in $S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$ can be regarded as particular symbols in Hörmander's classes $S(m,g) =: S^{\mu,\rho}$ corresponding to the weight function

$$m(x,\xi) = \langle x \rangle^\rho \langle \xi \rangle^\mu,$$

and to the slowly varying metric

$$g_{x,\xi} = \frac{|dx|^2}{\langle x \rangle^2} + \frac{|d\xi|^2}{\langle \xi \rangle^2},$$

cf. (1.1), so that for operators with symbols in those classes the standard pseudo-differential calculus works, see Hörmander [16], Chapter XVIII.

Now we recall that for classical symbols, i.e. symbols in $S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$, we have well defined symbol maps

$$\tilde{\sigma}^{\mu-k}_\psi \ : \ S^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \to r_e^{-\rho} \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1}), \tag{2}$$

$$\tilde{\sigma}^{\rho-j}_e \ : \ S^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \to r_\psi^{-\mu} \mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^n_+), \tag{3}$$

$$\tilde{\sigma}^{\mu-k,\rho-h}_{\psi,e} \ : \ S^{\mu,\rho}_{\mathrm{cl}(\xi,x)} \to \mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}), \tag{4}$$

for $k, j, h \in \mathbb{N}$. Precisely, given $A = \mathrm{Op}(a)$ with $a = (\mathrm{RC}^2)^* (r_\psi^{-\mu} r_e^{-\rho} a')$, $a' \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^n_+)$, we consider the Taylor expansion near $\mathbb{S}^n_+ \times \mathbb{S}^{n-1}$ of $a' \sim \sum_{i=0}^\infty r_\psi^i a_i$, $a_i \in \mathcal{C}^\infty(\mathbb{S}^n_+ \times \mathbb{S}^{n-1})$; then we define

$$\tilde{\sigma}^{\mu-k}_\psi (a) = r_e^{-\rho} a_k, \quad k \in \mathbb{N}.$$

Similarly, we expand $a' \sim \sum_{i=0}^{\infty} r_e^i b_i$ near $\mathbb{S}^{n-1} \times \mathbb{S}^n_+$, with $b_i \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^n_+)$, and we set

$$\tilde{\sigma}_e^{\rho-j}(a) = r_\psi^{-\mu} b_j, \quad j \in \mathbb{N}.$$

Finally, near $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1}$ we can write $a' \sim \sum_{i,j=0}^{\infty} r_\psi^j r_e^i c_{i,j}$, for suitable $c_{i,j} \in \mathcal{C}^\infty(\mathbb{S}^{n-1} \times \mathbb{S}^{n-1})$. This leads us to define

$$\tilde{\sigma}_{\psi,e}^{\mu-k,\rho-h}(a) = c_{h,k}, \quad h, k \in \mathbb{N}.$$

The results of this section will be established in terms of the symbol maps $\sigma_\psi^{\mu-k} := (\mathrm{RC}^* \times \mathrm{Id}) \circ \tilde{\sigma}_\psi^{\mu-k}$, $\sigma_e^{\rho-j} := (\mathrm{Id} \times \mathrm{RC}^*) \circ \tilde{\sigma}_e^{\rho-j}$, $\sigma_{\psi,e}^{\mu-k,\rho-h} := \tilde{\sigma}_{\psi,e}^{\mu-k,\rho-h}$.

Now we pass to consider trace functionals for some algebras of pseudo-differential operators, according to the following definition.

Definition 3. *Let $\mathcal{A}$ be a complex algebra; we say that a functional $L : \mathcal{A} \to \mathbb{C}$ is a* trace *if L is linear and vanishes on commutators.*

We recall that on the algebra $L^{-\infty}$ of the operators with kernels in $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ there is a unique trace functional which is given by

$$\mathrm{Tr}(A) = \int A(x,x)\, dx,$$

denoting by $A(x,y)$ the kernel of the operator A. Moreover, if we regard $L^{-\infty}$ as infinite matrix algebra (after choosing a base for $L^2(\mathbb{R}^n)$ given by functions in $\mathcal{S}(\mathbb{R}^n)$) we see that this trace is consistent with the usual trace of matrices. On the other hand it extends from $L^{-\infty}$ to $L_{\mathrm{cl}}^{-n-\epsilon,-n-\epsilon}$ for every $\epsilon > 0$: if $a \in S_{\mathrm{cl}(\xi,x)}^{-n-\epsilon,-n-\epsilon}$ then a^{w} is trace class and

$$\mathrm{Tr}(a^{\mathrm{w}}) = \iint a(x,\xi)\, dx\, d\xi.$$

To extend it further we need to regularize the resultant divergent integral; we do this using holomorphic families. Precisely, for $a \in S_{\mathrm{cl}(\xi,x)}^{\mu,\rho}$, $\mu, \rho \in \mathbb{Z}$, $\tau, z \in \mathbb{C} \times \mathbb{C}$, the function

$$\mathrm{Tr}(a_{\tau,z}^{\mathrm{w}}) = \iint \underbrace{a(x,\xi)[x]^\tau [\xi]^z}_{a_{\tau,z}(x,\xi)}\, dx\, d\xi \tag{5}$$

is well defined and holomorphic for $\mathrm{Re}\, z < -\mu - n$, $\mathrm{Re}\, \tau < -\rho - n$.

Theorem 4. *The function in* (5) *extends to a meromorphic function of* τ, z *with at most simple poles on the surfaces* $z = -\mu - n + j$, $\tau = -\rho - n + k$, $j, k \in \mathbb{N}$. *In particular, we have*

$$\mathrm{Tr}(a^{\mathrm{w}}_{\tau,z}) = \frac{1}{\tau z}\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) - \frac{1}{z}\widehat{\mathrm{Tr}}_{\psi}(a^{\mathrm{w}}) - \frac{1}{\tau}\widehat{\mathrm{Tr}}_{e}(a^{\mathrm{w}}) + \sum_{h+i\geq 2} c_{h,i}\tau^{h-1}z^{i-1}, \tag{6}$$

near $(\tau, z) = (0,0)$, *where the functionals* $\mathrm{Tr}_{\psi,e}$, $\widehat{\mathrm{Tr}}_{\psi}$ *and* $\widehat{\mathrm{Tr}}_{e}$, *defined as the residues in* (6), *have the following explicit expressions:*

$$\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) = (2\pi)^{-n}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \sigma^{-n,-n}_{\psi,e}(a)\, d\theta\, d\theta', \tag{7}$$

$$\begin{aligned}\widehat{\mathrm{Tr}}_{\psi}(a^{\mathrm{w}}) = (2\pi)^{-n}\lim_{\epsilon\to+\infty}\Bigg(&\int_{|x|\leq\epsilon}\int_{\mathbb{S}^{n-1}}\sigma^{-n}_{\psi}(a)\, d\theta\, dx \\ &-(2\pi)^n\log\epsilon\,\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) - \sum_{i=1}^{\rho+n}\frac{\epsilon^i}{i}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\sigma^{-n,i-n}_{\psi,e}(a)\, d\theta\, d\theta'\Bigg),\end{aligned} \tag{8}$$

$$\begin{aligned}\widehat{\mathrm{Tr}}_{e}(a^{\mathrm{w}}) = (2\pi)^{-n}\lim_{\epsilon\to+\infty}\Bigg(&\int_{|\xi|\leq\epsilon}\int_{\mathbb{S}^{n-1}}\sigma^{-n}_{e}(a)\, d\theta\, d\xi - (2\pi)^n\log\epsilon\,\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) \\ &-\sum_{i=1}^{\mu+n}\frac{\epsilon^i}{i}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}}\sigma^{i-n,-n}_{\psi,e}(a)\, d\theta\, d\theta'\Bigg).\end{aligned} \tag{9}$$

Remark 5. Let us note that the restrictions Tr_{ψ} and Tr_{e} of $\widehat{\mathrm{Tr}}_{\psi}$ and $\widehat{\mathrm{Tr}}_{e}$ to $\bigcup_{\mu\in\mathbb{Z}} L^{\mu,-n-1}_{\mathrm{cl}}$ and $\bigcup_{\rho\in\mathbb{Z}} L^{-n-1,\rho}_{\mathrm{cl}}$ are given by

$$\mathrm{Tr}_{\psi}(a^{\mathrm{w}}) = (2\pi)^{-n}\int_{\mathbb{R}^n_x}\int_{\mathbb{S}^{n-1}}\sigma^{-n}_{\psi}(a)\, d\theta\, dx, \qquad a\in\bigcup_{\mu\in\mathbb{Z}} S^{\mu,-n-1}_{\mathrm{cl}(\xi,x)}, \tag{10}$$

$$\mathrm{Tr}_{e}(a^{\mathrm{w}}) = (2\pi)^{-n}\int_{\mathbb{R}^n_\xi}\int_{\mathbb{S}^{n-1}}\sigma^{-n}_{e}(a)\, d\theta\, d\xi, \qquad a\in\bigcup_{\rho\in\mathbb{Z}} S^{-n-1,\rho}_{\mathrm{cl}(\xi,x)}, \tag{11}$$

and $\widehat{\mathrm{Tr}}_{\psi}$ and $\widehat{\mathrm{Tr}}_{e}$ turn out just the finite parts of the integrals in (10) and (11) when $a\in\bigcup_{\mu\in\mathbb{Z},\rho\in\mathbb{Z}} S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$.

Let us now set $\mathcal{I} := L^{-\infty}$ and $\mathcal{A} := \cup_{\rho\in\mathbb{Z}} \cup_{\mu\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}}/\mathcal{I}$,

$$\mathcal{I}_\psi = \bigcup_{\mu\in\mathbb{Z}} \bigcap_{\rho\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}}/\mathcal{I}, \quad \mathcal{I}_e = \bigcup_{\rho\in\mathbb{Z}} \bigcap_{\mu\in\mathbb{Z}} L^{\mu,\rho}_{\mathrm{cl}}/\mathcal{I} \tag{12}$$

$$\mathcal{A}_\psi = \mathcal{A}/\mathcal{I}_e, \quad \mathcal{A}_e = \mathcal{A}/\mathcal{I}_\psi, \quad \mathcal{A}_{\psi,e} = \mathcal{A}/(\mathcal{I}_\psi + \mathcal{I}_e). \tag{13}$$

The following result holds.

Theorem 6. *The functional* $\mathrm{Tr}_{\psi,e}$ *defines a trace on the algebra* $\mathcal{A}$ *which vanishes on* $\mathcal{I}_\psi$ *and* $\mathcal{I}_e$ *and therefore it induces traces on* $\mathcal{A}_\psi, \mathcal{A}_e$ *and* $\mathcal{A}_{\psi,e}$*. On* $\mathcal{I}_\psi$ *and* $\mathcal{I}_e$ *trace functionals are given respectively by* Tr_ψ *and* Tr_e*.*
For all these algebras, the above functionals are the unique traces up to multiplication by a constant.

It is interesting to note that the above traces coincide, up to a multiplicative constant, with some Dixmier traces, cf. [11], we are going to define. Let $\mathbb{K}$ be the ideal of compact operators on a separable Hilbert space H. For $T \in \mathbb{K}$, let $\mu_n(T)$ be the sequence of eigenvalues of $|T|$; $\sigma_N(T) = \sum_{n=0}^{N} \mu_n(T),\ N \in \mathbb{N}$.

Definition 7. *We define the ideals of compact operators*

$$\mathcal{L}^{(1,\infty)}(H) = \left\{ T \in \mathbb{K} : \left(\frac{\sigma_N(T)}{\log N} \right)_{N\in\mathbb{N}} \in l^\infty(\mathbb{N}) \right\}$$
$$\mathcal{L}^{(1,\infty)}_{\log}(H) = \left\{ T \in \mathbb{K} : \left(\frac{\sigma_N(T)}{(\log N)^2} \right)_{N\in\mathbb{N}} \in l^\infty(\mathbb{N}) \right\}.$$

The Dixmier traces

$$\mathrm{Tr}_\omega : \mathcal{L}^{(1,\infty)}(H) \to \mathbb{C},\ \mathrm{Tr}_\omega(T) = \lim_\omega \left(\frac{\sigma_N(T)}{\log N} \right)$$
$$\mathrm{Tr}'_\omega : \mathcal{L}^{(1,\infty)}_{\log}(H) \to \mathbb{C},\ \mathrm{Tr}'_\omega(T) = \lim_\omega \left(\frac{\sigma_N(T)}{(\log N)^2} \right)$$

are hence defined first for $T \geq 0$*, then extended by linearity.*

In Definition 7 we used the notation $\lim_\omega$ for Connes' ω-limit, see Connes [4] for the precise definition. Here we only remark that it is well defined on bounded sequences and coincides with the usual limit on convergent sequences.

The main features of Dixmier traces are listed here.

- If $T \geq 0$ then $\mathrm{Tr}_\omega(T) \geq 0$.
- If S is any bounded operator and $T \in \mathcal{L}^{(1,\infty)}(H)$, then $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$.

• $\mathrm{Tr}_\omega(T)$ is independent of the choice of the inner product on H, i.e. it only depends on the Hilbert space H as a topological vector space.
• Tr_ω vanishes on trace class operators.

Analogous remarks hold for Tr'_ω. Now we can state the above mentioned result.

Theorem 8. *The following inclusions hold:*

$$\begin{aligned} &L^{-n,-n}_{\mathrm{cl}} \subset \mathcal{L}^{(1,\infty)}_{\log}(L^2(\mathbb{R}^n)),\\ &L^{-n,\rho}_{\mathrm{cl}} \subset \mathcal{L}^{(1,\infty)}(L^2(\mathbb{R}^n)) \quad \text{if } \rho < -n,\\ &L^{\mu,-n}_{\mathrm{cl}} \subset \mathcal{L}^{(1,\infty)}(L^2(\mathbb{R}^n)) \quad \text{if } \mu < -n. \end{aligned}$$

Furthermore we have

$$\begin{aligned} &\mathrm{Tr}_{\psi,e}(a^{\mathrm{w}}) = 2n^2\mathrm{Tr}'_\omega(a^{\mathrm{w}}) \quad \text{for } a \in S^{-n,-n}_{\mathrm{cl}(\xi,x)},\\ &\mathrm{Tr}_{\psi}(a^{\mathrm{w}}) = n\mathrm{Tr}_\omega(a^{\mathrm{w}}) \quad \text{for } a \in S^{-n,\rho}_{\mathrm{cl}(\xi,x)},\ \rho\in\mathbb{Z}, \rho<-n,\\ &\mathrm{Tr}_{e}(a^{\mathrm{w}}) = n\mathrm{Tr}_\omega(a^{\mathrm{w}}) \quad \text{for } a \in S^{\mu,-n}_{\mathrm{cl}(\xi,x)},\ \mu\in\mathbb{Z}, \mu<-n. \end{aligned}$$

Theorem 8 is analogous to the result established by Connes [3] for classical operators on compact manifolds. The proof given in [23] relies on the asymptotic distribution of eigenvalues of a formally self-adjoint elliptic pseudo-differential operators in our classes. Indeed, after reducing by linearity to the case of an elliptic operator $A = a^{\mathrm{w}} > 0$, with Weyl symbol $a > 0$, one can deduce estimates for the eigenvalues of A from estimates for the counting function $N(\lambda)$ of the operator A^{-1}. In this connection, we have the following formula of Maniccia [18], Maniccia-Panarese [19], which can be also deduced by Hörmander's Weyl formula in [15] by computing the symplectic volume of the set $\{(x,\xi)\in\mathbb{R}^{2n} : a(x,\xi)\le\lambda\}$.

Theorem 9. *Let $a \in S^{\mu,\rho}_{\mathrm{cl}(\xi,x)}$, $\mu>0,\rho>0$, be a positive elliptic symbol and denote by $N(\lambda)$ the counting function associated with the operator a^{w}. Then for every $0<\delta_1<\frac{2}{3\rho}, 0<\delta_2<\frac{2}{3\mu}$, we have*

$$N(\lambda) = \begin{cases} C_\mu\lambda^{\frac{n}{\mu}}\log\lambda + O\left(\lambda^{\frac{n}{\mu}}\right) & \text{for } \mu=\rho,\\ C'_\mu\lambda^{\frac{n}{\mu}} + O\left(\lambda^{\frac{n}{\mu}-\delta_1}\right) & \text{for } \mu<\rho,\\ C''_\rho\lambda^{\frac{n}{\rho}} + O\left(\lambda^{\frac{n}{\rho}-\delta_2}\right) & \text{for } \mu>\rho, \end{cases}$$

where

$$C_\mu = \frac{(2\pi)^{-n}}{n\mu}\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-1}} \sigma^{\mu,\mu}_{\psi,e}(a)^{-\frac{n}{\mu}}\, d\theta\, d\theta',$$
$$C'_\mu = \frac{(2\pi)^{-n}}{n}\int_{\mathbb{R}^n_x}\int_{\mathbb{S}^{n-1}} \sigma^{\mu}_{\psi}(a)^{-\frac{n}{\mu}}\, d\theta\, dx,$$
$$C''_\rho = \frac{(2\pi)^{-n}}{n}\int_{\mathbb{R}^n_\xi}\int_{\mathbb{S}^{n-1}} \sigma^{\rho}_{e}(a)^{-\frac{n}{\rho}}\, d\theta\, d\xi.$$

We finally address to Nicola [24] for the study of other invariants, namely the K-theory groups of the subalgebra of the SG operators of order zero.

3. Essential spectrum

We are interested in locating the essential spectrum of pseudo-differential operators with symbol in the class $S^{0,0}$, see (1) and Remark 2 for the definition. Here the essential spectrum $\Sigma_e(A)$ of a bounded operator A on a Hilbert space is defined as the set of all complex numbers λ such that $A-\lambda I$ is not Fredholm with zero index. Since a symbol $a\in S^{0,0}$ in particular satisfies

$$\lim_{|x|\to+\infty}\sup_{\xi\in\mathbb{R}^n}\langle\xi\rangle^{|\beta|}|\partial^\alpha_x\partial^\beta_\xi a(x,\xi)| = 0, \quad |\alpha|\neq 0,$$

a result due to Wong [31] applies and gives the estimate

$$\Sigma_e(a^{\mathrm{w}}) \subset \{\lambda\in\mathbb{C} : |\lambda| \le \sup_{(x,\xi)\in\mathbb{R}^{2n}} |a(x,\xi)|\}. \tag{14}$$

We observe that actually a sharper result holds:

Theorem 10. *Let $a\in S^{0,0}$. We have*

$$\Sigma_e(a^{\mathrm{w}}) \subset \{\lambda\in\mathbb{C} : |\lambda| \le \limsup_{(x,\xi)\to\infty} |a(x,\xi)|\}.$$

Indeed, cutting the symbol away from the ball $B(0,R)$ we modify the operator a^w with a compact perturbation, which does not have effect on the essential spectrum. Then letting $R\to+\infty$ we obtain Theorem 10 (for the more general classes considered by Wong) from (14).

Here we want to give another proof of Theorem 10 based on some elementary properties of the so-called localization operators studied, among others, by Wong [32],[33].

Precisely, let G be a topological locally compact Hausdorff group with left Haar measure μ; let X be a Hilbert space, $\mathbb{B}$ ($\mathbb{K}$ resp.) the algebra of all bounded (compact resp.) operators on X. Let $\pi : G \to \mathbb{B}$ be an irreducible square-integrable unitary representation of G and, finally, let $\phi \in X$ be an admissible wavelet; set $c_\phi = \int_G |(\phi, \pi(g)\phi)_H|^2 \, d\mu(g)$. With any $F \in L^\infty(G)$ we associate the operator $L_F : X \to X$, defined for $x \in X$ by

$$L_F x = \frac{1}{c_\phi} \int_G F(g)(x, \pi(g)\phi)_H \pi(g)\phi \, d\mu(g). \tag{15}$$

Then L_F is a bounded linear operator on X and

$$\|F\|_{\mathbb{B}} \leq \|F\|_{L^\infty(G)}. \tag{16}$$

The proof of Theorem 10 requires the following proposition which may be of intrinsic interest. Here we consider the so-called Alexandrov compactification $G_\infty := G \cup \{\infty\}$ of G; it is a compact Hausdorff space.

Proposition 11. *Let $F \in L^\infty(G)$ and L_F as in* (15). *Then*

$$\inf_{K \in \mathbb{K}} \|L_F + K\|_{\mathbb{B}} \leq \limsup_{g \to \infty} |F(g)|.$$

P r o o f. Let $\mathcal{U}$ be the family of all open neighborhoods of ∞ in G_∞. By the very definition, if $U \in \mathcal{U}$ then $G_\infty \setminus U$ is a compact subspace of G. For $U \in \mathcal{U}$ we denote by χ_U the characteristic function of U and we set $F_U := (1 - \chi_U)F$, as a function on G. Since $F_U \in L^1(G)$ we have that L_{F_U} is trace class and therefore compact on X, see Proposition 3.4 by Wong [33].

Then for every $U \in \mathcal{U}$ by (16) we have

$$\inf_{K \in \mathbb{K}} \|L_F + K\|_{\mathbb{B}} \leq \|L_F - L_{F_U}\|_{\mathbb{B}} \leq \|F - F_U\|_{L^\infty(G)} = \sup_{g \in U \setminus \{\infty\}} |F(g)|.$$

Now $\inf_{U \in \mathcal{U}} \sup_{g \in U \setminus \{\infty\}} |F(g)| = \limsup_{g \to \infty} |F(g)|$, so that the proposition is proved. $\square$

P r o o f o f T h e o r e m 1 0. By standard results of Functional Analysis, as in the proof given by Wong [31] it suffices to prove that

$$\inf_{K \in \mathbb{K}} \|a^{\mathrm{w}} + K\|_{\mathbb{B}} \leq \limsup_{(x,\xi) \to \infty} |a(x, \xi)|. \tag{17}$$

Let us consider the Weyl-Heisenberg group $(WH)^n = \mathbb{C}^n \times \mathbb{R}/2\pi\mathbb{Z}$. We apply Proposition 11 with $G = (WH)^n$, $X = L^2(\mathbb{R}^n)$, $(\pi(q,p,t)f)(x) = e^{i(px - qp + t)} f(x - q)$ for $x \in \mathbb{R}^n$, $\phi(x) = \pi^{-n/4} e^{-|x|^2/2}$ and $F = a$, so that (17) holds for L_a. On the other hand, setting $\Lambda(x,y) = \pi^{-n} e^{-|x|^2 - |y|^2}$ it is easy to see that $L_a = (a \star \Lambda)^{\mathrm{w}} = a^{\mathrm{w}} + \tilde{a}^{\mathrm{w}}$, where $\tilde{a} \in S^{-1,-1}$. Since $\tilde{a}^{\mathrm{w}}$ is compact, (17) follows for a^{w}. $\square$

4. SG hyperbolicity

We fix here attention on operators in $]-T,T[\times\mathbb{R}^n$ of the form

$$P = D_t^m + \sum_{j=1}^{m} P_j D_t^{m-j} \tag{18}$$

where $P_j = P_j(t,x,D_x)$ is a family of SG pseudo-differential operators, with symbols $p_j(t,x,\xi) \in C^\infty(]-T,T[, S^{j,j})$, $j = 1,\dots,m$. In the following definition, cf. Coriasco [7], we write that $\tilde{p}_j$ is a principal symbol of p_j to mean that $p_j - \tilde{p}_j \in C^\infty(]-T,T[, S^{j-1,j-1})$.

Definition 12. *The operator P in* (18) *is called* SG *hyperbolic with constant multiplicity, if we may choose principal symbols $\tilde{p}_j$ so that the characteristic equation*

$$\lambda^m + \sum_{j=1}^{m} \tilde{p}_j(t,x,\xi)\lambda^{m-j} = 0 \tag{19}$$

has $M \le m$ distinct real roots $\lambda_h = \lambda_h(t,x,\xi) \in C^\infty(]-T,T[, S^{1,1})$, $h = 1,\dots,M$, which satisfy for positive constants C and R:

$$\lambda_{h+1}(t,x,\xi) - \lambda_h(t,x,\xi) \ge C\langle x\rangle\langle \xi\rangle, \tag{20}$$

$h = 1,\dots,M-1$, $|x| + |\xi| \ge R$. The operator P is called strictly hyperbolic if $M = m$, i.e. the multiplicity of all the roots λ_h is equal 1.

Let us then consider the case when $p_j(t,x,\xi) \in C^\infty(]-T,T[, S^{j,j}_{\mathrm{cl}(\xi,x)})$, $j = 1,\dots,m$. We may obviously reset Definition 12 requiring $\lambda_h \in C^\infty(]-T,T[, S^{1,1}_{\mathrm{cl}(\xi,x)})$ and $p_j - \tilde{p}_j \in C^\infty(]-T,T[, S^{j-1,j-1}_{\mathrm{cl}(\xi,x)})$. Fix attention on $\sigma^j_\psi(p_j)$, $\sigma^j_e(p_j)$, $\sigma^{j,j}_{\psi,e}(p_j)$; it is natural to consider, instead of (19), the characteristic equations:

$$\lambda^m + \sum_{j=1}^{m} \sigma^j_\psi(p_j)(t,x,\xi)\lambda^{m-j} = 0, \quad \xi \ne 0, \tag{21}$$

$$\lambda^m + \sum_{j=1}^{m} \sigma^j_e(p_j)(t,x,\xi)\lambda^{m-j} = 0, \quad x \ne 0, \tag{22}$$

$$\lambda^m + \sum_{j=1}^{m} \sigma^{j,j}_{\psi,e}(p_j)(t,x,\xi)\lambda^{m-j} = 0, \quad x \ne 0,\ \xi \ne 0. \tag{23}$$

Proposition 13. *Let P be defined as in* (18) *with* $p_j \in C^\infty(]-T,T[, S^{j,j}_{\mathrm{cl}(\xi,x)})$, $j = 1,\dots,m$. *Then P is* SG *hyperbolic with constant multiplicity if and only if* $(4.4), (4.5), (4.6)$ *have $M \le m$ real distinct roots, which we denote by* λ^ψ_h, *and respectively* λ^e_h, $\lambda^{\psi,e}_h$, $h = 1,\dots,M$.

P r o o f. Assume that P satisfies Definition 12 with $\lambda_h \in C^\infty(]-T,T[, S^{1,1}_{\mathrm{cl}(\xi,x)})$. Since $\sigma^j_\psi(\tilde{p}_j) = \sigma^j_\psi(p_j)$, using the properties of the σ_ψ-symbols (see for example Section 1.4.3 in Schulze [27]) we obtain from (19):

$$\sigma^m_\psi(\lambda^m_h + \sum_{j=1}^m \tilde{p}_j \lambda_h^{m-j}) = (\sigma^1_\psi(\lambda_h))^m + \sum_{j=1}^m \sigma^j_\psi(p_j)(\sigma^1_\psi(\lambda_h))^{m-j} = 0. \tag{24}$$

Hence $\lambda^\psi_h = \sigma^1_\psi(\lambda_h)$, and similarly $\lambda^e_h = \sigma^1_e(\lambda_h)$, $\lambda^{\psi,e}_h = \sigma^{1,1}_{\psi,e}(\lambda_h)$, $h = 1,\dots,M$, satisfy the requirements of Proposition 13.

In the opposite direction, standard arguments for hyperbolic equations, cf. Cordes [5], show that the distinct roots λ^ψ_h, λ^e_h, $\lambda^{\psi,e}_h$ of $(4.4), (4.5), (4.6)$ respect. are C^∞ functions of t, x, ξ, homogeneous of degree 1 respect. in ξ, x, and (x,ξ) separately. Since $\sigma^j_e \sigma^j_\psi = \sigma^j_\psi \sigma^j_e = \sigma^{j,j}_{\psi,e}$, we may argue as in (24) to pass from (21), (22) to (23); after possibly re-ordering the roots, this gives

$$\lambda^{\psi,e}_h = \sigma^1_e(\lambda^\psi_h) = \sigma^1_\psi(\lambda^e_h), \quad h = 1,\dots,M.$$

Following the proof of Proposition 1.4.28 in Schulze [27], we then define

$$\lambda_h(t,x,\xi) = \omega(\xi)\lambda^\psi_h(t,x,\xi) + \omega(x)\lambda^e_h(t,x,\xi) + \omega(x)\omega(\xi)\lambda^{\psi,e}_h(t,x,\xi) \tag{25}$$

where $\omega \in C^\infty(\mathbb{R}^n)$ is non-negative, vanishing in a neighborhood of the origin and identically equal to 1 outside a ball centered in the origin; we have $\lambda_h \in C^\infty(]-T,T[, S^{1,1}_{\mathrm{cl}(\xi,x)})$ with $\sigma^1_\psi(\lambda_h) = \lambda^\psi_h$, $\sigma^1_e(\lambda_h) = \lambda^e_h$, $\sigma^{1,1}_{\psi,e}(\lambda_h) = \lambda^{\psi,e}_h$.

Let us check that the λ_h in (25) satisfy Definition 12. In fact, assuming $M = m$ for simplicity of notation, we have

$$\prod_{h=1}^m (\lambda - \lambda_h) = \lambda^m + \sum_{j=1}^m \tilde{p}_j \lambda^{m-j}$$

where we can express $\tilde{p}_j \in C^\infty(]-T,T[, S^{j,j}_{\mathrm{cl}(\xi,x)})$ in terms of λ_h by the Cardano-Viète identities:

$$\tilde{p}_1 = -\sum \lambda_h,\ \tilde{p}_2 = \sum \lambda_h\lambda_k, \dots,\ \tilde{p}_m = (-1)^m \lambda_1 \dots \lambda_m.$$

We then obtain from the same identities applied to the roots of (21):

$$\sigma^1_\psi(\tilde{p}_1) = \sigma^1_\psi\left(-\sum \lambda_h\right) = -\sum \sigma^1_\psi(\lambda_h) = -\sum \lambda_h^\psi = \sigma^1_\psi(p_1),$$

$$\ldots\ldots$$

$$\sigma^m_\psi(\tilde{p}_m) = (-1)^m\sigma^m_\psi(\lambda_1\ldots\lambda_m) = (-1)^m\sigma^1_\psi(\lambda_1)\ldots\sigma^1_\psi(\lambda_m) = \\ = (-1)^m\lambda_1^\psi\ldots\lambda_m^\psi = \sigma^m_\psi(p_m),$$

and similarly $\sigma^j_e(\tilde{p}_j) = \sigma^j_e(p_j)$, $\sigma^{j,j}_{\psi,e}(\tilde{p}_j) = \sigma^{j,j}_{\psi,e}(p_j)$ for $j = 1,\ldots,m$. This shows that $\tilde{p}_j$ are principal symbols of p_j, giving in (19) the roots λ_h.

Finally, if we define $\tau_{hk} = \lambda_h - \lambda_k$, from our assumption we have $\sigma^1_\psi(\tau_{hk}) = \lambda_h^\psi - \lambda_k^\psi \neq 0$ for $\xi \neq 0$, $\sigma^1_e(\tau_{hk}) = \lambda_h^e - \lambda_k^e \neq 0$ for $x \neq 0$, $\sigma^{1,1}_{\psi,e}(\tau_{hk}) = \lambda_h^{\psi,e} - \lambda_k^{\psi,e} \neq 0$ for $x \neq 0$, $\xi \neq 0$. According to Proposition 1.4.37 in Schulze [27], this implies the ellipticity of τ_{hk}, namely $|\tau_{hk}(t,x,\xi)| \geq C\langle x\rangle\langle\xi\rangle$ for $|x| + |\xi| \geq R$. Hence, possibly after re-ordering the roots λ_h, we deduce (20). □

We recall now the result of Coriasco [7], limiting here attention for the sake of simplicity to the case of one multiple characteristic $\lambda = \lambda(t,x,\xi) \in C^\infty(]-T,T[,S^{1,1})$. We may then re-write our operator in the factorized form

$$(26) \qquad P = (D_t - \lambda(t,x,D_x))^m + \sum_{j=1}^m a_j(t,x,D_x)(D_t - \lambda(t,x,D_x))^{m-j}$$

with $a_j(t,x,\xi) \in C^\infty(]-T,T[,S^{j,j})$. Consider the Cauchy problem

$$(27) \qquad Pu = f(t,x),\ D_t^s u|_{t=0} = g_s(x), \quad s = 0,\ldots,m-1.$$

Definition 14. *We say that P in* (26) *satisfies the* SG *Levi condition if $a_j \in C^\infty(]-T,T[,S^{0,0})$ for all $j = 1,\ldots,m$. Given $p,q \in [0,1[$ with $p+q < 1$, we say that P satisfies the (p,q) Levi condition if $a_j \in C^\infty(]-T,T[,S^{pj,qj})$ for $j = 1,\ldots,m$.*

Theorem 15. *Let P in* (26) *satisfy the* SG *Levi condition. Then the Cauchy problem* (27) *is well-posed in $\mathcal{S}(\mathbb{R}^n)$, namely for every $f \in C^\infty(]-T,T[,\mathcal{S}(\mathbb{R}^n))$, $g_s \in \mathcal{S}(\mathbb{R}^n)$, $s = 0,\ldots,m-1$, there exists a unique solution $u \in C^\infty(]-T,T[,\mathcal{S}(\mathbb{R}^n))$.*

Same statement is valid in $\mathcal{S}'(\mathbb{R}^n)$. The solution $u(t,x)$ can be expressed in terms of SG-Fourier integral operators, see Coriasco [6],[7], Coriasco-Rodino [9], Coriasco-Panarese [8]. Coriasco-Maniccia [10] prove for $u(t,x)$ propagation of the related wave front set ($\mathcal{S}$-wave front set, in the terminology of the authors).

To treat the case when the SG Levi condition is replaced by the weaker (p, q)-Levi condition, it is convenient to refer to the spaces $S^\theta_\theta(\mathbb{R}^n)$ of Gelfand-Shilov [13]. We recall that $f \in S^\theta_\theta(\mathbb{R}^n)$, $\theta > 1$, if for positive constants a and B

$$|D^\beta u(x)| \leq B^{|\beta|+1}(\beta!)^\theta e^{-a|x|^{1/\theta}}.$$

Under suitable analytic estimates for $\lambda(t, x, \xi)$ and $a_j(t, x, \xi)$ in (26), Cappiello [1],[2] proved the following result.

Theorem 16. *Let the* (p, q) *Levi condition be satisfied and assume* $p + q < 1/\theta < 1$. *Then the Cauchy problem* (27) *is well-posed in* $S^\theta_\theta(\mathbb{R}^n)$ *and in the dual* $S^{\theta\prime}_\theta(\mathbb{R}^n)$.

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